

# Broadband macromodelling of passive components using orthonormal vector fitting

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Vector fitting is widely accepted as a robust macromodelling tool for efficient frequency domain identification of passive components. The orthonormal vector fitting technique is introduced, which improves the numerical stability of the method, by using orthonormal rational functions. This leads to better conditioned equations, reduces the numerical sensitivity to the choice of starting poles significantly, limits the number of required iterations, and reduces the overall computation time.

**Introduction:** Compact rational macromodels, based on measurements or full-wave EM simulations, are very important for efficient time domain and frequency domain simulation of high-speed interconnection structures. Gustavsen and Semlyen proposed vector fitting (VF) as an accurate iterative macromodelling technique [1]. In [2], this approach was recognised as a reformulation of a Sanathanan-Koerner (SK) iteration [3], which starts from an initial guess of model parameters and relocates them to obtain an optimal fit. The robustness of the method is mainly due to the use of rational bases instead of polynomials, which are numerically advantageous when the initial poles are properly chosen. The new OVF technique orthonormalises these partial fractions, in order to improve the numerical properties of the method even further. This leads to better conditioned equations, reduces the numerical sensitivity of the model parameterisation to the choice of starting poles significantly, limits the number of required iterations, and reduces the overall macromodelling time. Afterwards, the rational models can easily be realised as a compact SPICE circuit.

**Rational fitting technique: vector fitting:** The frequency domain behaviour of passive linear electrical components can be modelled by a rational function model:

$$R(s_k) = \frac{N(s_k)}{D(s_k)} = \frac{\sum_{n=0}^N N_n \varphi_n(s_k)}{\sum_{d=0}^D D_d \varphi_d(s_k)} \simeq H(s_k) \quad (1)$$

$R(s_k)$  approximates the simulated data samples  $H(s_k)$  in a least squares sense, at the discrete complex frequencies  $s_k, \forall k=0, \dots, K$ .  $N_n$  and  $D_d$  are the unknown system parameters, and  $N$  and  $D$  are the order of numerator and denominator, respectively.

The basis functions  $\varphi_n(s)$  can be polynomial bases, such as power series basis or orthogonal Chebyshev polynomials, or they can also be rational functions. Vector fitting proposed the use of simple partial fractions as basis functions in the numerator and denominator, which share a common set of poles. The denominator has an additional basis function, which equals the constant value 1. Given an initial set of starting poles (iteration step 0), or a previous estimate of the poles (iteration step  $t-1$ ), the poles and zeros of iteration step  $t$  are relocated by minimising the following linear least-squares problem iteratively ( $D_0=1$ ):

$$\arg \min_{D^{(t)}, N^{(t)}} \sum_{k=0}^K \frac{1}{\left| \sum_{d=0}^D D_d^{(t-1)} \varphi_d(s_k) \right|^2} \times \left| \sum_{d=0}^D D_d^{(t)} \varphi_d(s_k) (H(s_k) - \sum_{n=0}^N N_n^{(t)} \varphi_n(s_k)) \right|^2 \quad (2)$$

After simplification, the common poles cancel out and a ratio of two polynomials is obtained. In practice, the zeros of  $(\sum_{d=0}^D D_d^{(t)} \varphi_d(s_k))$  are calculated, since they become the new poles (say  $-a_p$ ) of the next (or final) iteration. This can easily be done by calculating a minimal state-space realisation, and solving an eigenvalue problem, as shown in Appendix B of [1]. Once the poles are known, the residues of the partial fraction expansion of (1) can be estimated by solving

$$\arg \min_C \sum_{k=0}^K \left| \left( \sum_{p=1}^P \frac{c_p}{s_k + a_p} \right) - H(s_k) \right|^2 \quad (3)$$

such that the fitting error is minimised. An optional constant and linear term can be added to the partial fraction expansion, to vary the order of

the numerator. Bounded-input-bounded-output (BIBO) system stability can be enforced by flipping poles into the left half plane.

**Orthonormal basis functions:** Instead of using the partial fractions as rational basis functions, it is shown that orthonormal rational basis functions can lead to significant improvements in numerical conditioning [4]. A straightforward way to calculate an orthonormal basis is to apply a Gram-Schmidt procedure on the partial fractions [5]. Hence, orthonormal rational functions  $\phi_p(s)$  are obtained, which are linear combinations of the partial fractions  $\varphi_p(s)$ , of the form

$$\phi_p(s) = \frac{Q_p(s)}{\prod_{j=1}^p (s + a_j)} \quad (4)$$

for  $p=1, \dots, P$  and  $Q_p(s)$  an arbitrary polynomial of order  $p-1$ . The inner product is defined as

$$\langle \phi_i(s), \phi_j(s) \rangle = \frac{1}{2\pi i} \int_{\text{j}\mathbb{R}} \phi_i(s) \phi_j^*(s) ds \quad (5)$$

The  $Q_p(s)$  polynomial is determined by applying the orthonormality conditions on the basis functions, and so the following basis functions are obtained

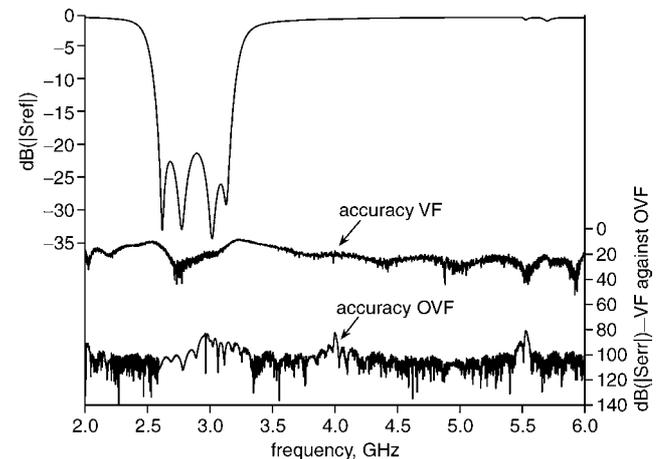
$$\phi_p(s) = \kappa_p \left( \prod_{j=1}^{p-1} \frac{s - a_j^*}{s + a_j} \right) \frac{\sqrt{2\text{Re}(a_p)}}{s + a_p} \quad (6)$$

where  $\kappa_p$  represents an arbitrary unimodular complex number. To enforce real-valuedness of the transfer function, the basis functions corresponding to a complex conjugate pair are made real-valued. This is done by forming a linear combination of  $\phi_i(s)$  and  $\phi_{i+1}(s)$  [6]. This way, two orthonormal functions of the following form are obtained

$$\phi_i(s) = \sqrt{a_i + a_{i+1}} \left( \prod_{j=1}^{i-1} \frac{s - a_j^*}{s + a_j} \right) \frac{s - \sqrt{a_i a_{i+1}}}{(s + a_i)(s + a_{i+1})} \quad (7)$$

$$\phi_{i+1}(s) = \sqrt{a_i + a_{i+1}} \left( \prod_{j=1}^{i-1} \frac{s - a_j^*}{s + a_j} \right) \frac{s + \sqrt{a_i a_{i+1}}}{(s + a_i)(s + a_{i+1})} \quad (8)$$

After parameterisation of the system parameters using the Sanathanan-Koerner iteration [3], the poles can be calculated by solving a similar eigenvalue problem. The minimal state-space realisation of  $\sum_{d=0}^D D_d^{(t)} \varphi_d(s_k)$  can accurately be obtained by cascading a number of smaller, first- or second-order sections [7].



**Fig. 1** Magnitude fitting error against frequency  
Model calculated using 100 samples and real poles in single iteration

**Example: lowpass filter:** The reflection coefficient  $S_{11}$  of a lowpass filter was modelled over the frequency range of interest [2–6 GHz], based on a set of 20 equidistant support samples. The magnitude of the reference data is shown in Fig. 1 (see left axis). The initial poles are chosen as complex conjugate pairs with a fixed real part, and an imaginary part, which is linearly distributed over the frequency range of interest. For this example, the frequencies are scaled by  $10^9$ . To avoid convergence issues caused by the SK-linearisation, the number of poles is chosen equal to the number of samples to obtain an

interpolating function. Both techniques (VF and the new OVF) are used to fit the rational data using the same set of initial poles, and one single iteration is allowed. Table 1 shows the condition number of both techniques (ratio between largest and smallest singular value), and the RMS error in the fitted data points against initial pole location. Clearly, the conditioning of OVF is remarkably superior to the conditioning of classical VF, and the RMS error is significantly smaller. The computation time of one single iteration is approximately the same for both techniques. However, OVF outperforms classical VF since fewer iterations are needed. Note, however, that best results are obtained when the initial poles are chosen as suggested in [1].

**Table 1:** Condition number and RMS error against initial pole location

$\Re(-a_p)$	Cond. VF	Cond. OVF	RMS VF	RMS OVF
-4	$1.1964 \times 10^{17}$	$2.3676 \times 10^{10}$	$1.64 \times 10^{-1}$	$9.48 \times 10^{-11}$
-3	$1.9803 \times 10^{15}$	$1.4357 \times 10^9$	$5.24 \times 10^{-2}$	$7.99 \times 10^{-12}$
-2	$3.1673 \times 10^{12}$	$5.1195 \times 10^7$	$1.44 \times 10^{-5}$	$2.22 \times 10^{-13}$
-1	$1.4154 \times 10^9$	$1.2351 \times 10^6$	$3.72 \times 10^{-10}$	$6.45 \times 10^{-15}$

When the initial poles are chosen real and linearly spaced over the frequency interval, the problem is known to be quite ill-conditioned [1]. If 100 data samples are fitted (using 100 poles), OVF needed five iterations to obtain an RMS error of  $3.08 \times 10^{-15}$ , whereas VF needed 13 iterations to obtain a similar accuracy of  $2.52 \times 10^{-15}$ . Hence, in this situation, the computation time of OVF is only 5/13 ( $\approx 38, 46\%$ ) of the computation time of VF. As an illustration, Fig. 1 shows the overall fitting error after the first iteration (see right axis).

*Conclusions:* This Letter enhances the numerical properties of vector fitting by means of orthonormal rational functions. Owing to the

improvements in numerical conditioning, accurate broadband macromodels are often obtained in fewer iterations. This leads to a significant reduction of the overall macromodelling time, especially when the initial set of poles is not chosen optimally.

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