A Discussion of "Rational Approximation of Frequency Domain Responses by Vector Fitting"

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***Abstract*—*Vector fitting* (VF) is a popular iterative rational approximation technique for sampled data in the frequency domain. VF is nowadays widely investigated and used in the *Power Systems* and *Microwave Engineering* communities. The VF methodology is recognized as an elegant version of the Sanathanan–Koerner iteration with a well-chosen basis.**

***Index Terms*—Rational interpolation, system identification, vector fitting.**

I. Introduction

Broadband rational approximations of the admittance matrix Y are of paramount importance for accurate transient simulation of the frequency-dependent behavior of linear power systems, such as transmission lines and transformers. The compact rational pole-zero models can easily be used in EMTP-type programs, and they can replace the tedious and error-prone numerical transient simulations by fast and efficient iterative convolutions.

In [1], a new rational interpolation technique, *vector fitting* (VF), was introduced. References [2]–[4] provide more information on the subject. In this letter, we elaborate on the internals of this technique. A thorough mathematical analysis provides insight into the inner working of the VF algorithm and teaches how it can be improved, e.g., by using an orthonormal rational basis [5]. Throughout this letter, we have adopted the same notation as in [1]. We assume that the reader is familiar with the VF methodology.

In Section II, we discuss the pole-based basis that is used in VF. Section III describes a general rational iterative least-squares framework. Section IV finally provides the link with the VF methodology. Section V contains our conclusions.

II. Preliminaries: the Pole-Based Basis

We fix degree D and fix $\overline{a}_1, \ldots, \overline{a}_D \in \mathbb{C}$ (the *starting poles*) and look at the basis functions

$$f_i : \mathbb{C} \to \mathbb{C} : s \mapsto \frac{1}{s - \overline{a}_i} \text{ for } i = 1, \ldots, D \tag{1}$$

where $s = j\omega$. If we form linear combinations of these basis functions and add a polynomial $L(s)$ of degree E, we always get fractions with the same denominator

$$\begin{aligned}\sum_{i=1}^{D} \alpha_i f_i(s) + L(s) &= \frac{P(s)}{Q(s)} \\ &= \frac{\sum_{i=1}^{D} \alpha_i \left(\prod_{j\neq i}(s - \overline{a}_j)\right) + L(s)\prod_{i=1}^{D}(s - \overline{a}_i)}{\prod_{i=1}^{D}(s - \overline{a}_i)}\end{aligned} \tag{2}$$

where we have introduced the abbreviations $P(s)$ for the numerator (which is of degree $E + D$; we take $E = -1$ when $L(s) = 0$) and $Q(s)$ for the denominator.

On the other hand, if we have a polynomial $P'(s)$ of degree $E' + D$, we can always write

$$\frac{P'(s)}{Q(s)} = \sum_{i=1}^{D} \beta_i f_i(s) + L'(s) \tag{3}$$

for some β_i and some polynomial $L'(s)$ of degree E'. This is nothing more than the partial fraction expansion.

III. Preliminaries: The Iterative Least-Squares Scheme

Suppose we want to approximate the values $H_1, \ldots, H_n \in \mathbb{C}$ at sample points $s_1, \ldots, s_n \in \mathbb{C}$ by a model of the form

$$\tilde{H}(s) = \frac{\sum_{i=1}^{N} \alpha_i f_i(s)}{\sum_{j=1}^{D} \beta_j g_j(s)} = \frac{p(s,\alpha)}{q(s,\beta)} \tag{4}$$

for some fixed *basis functions* f_i and g_i. An optimal solution (in the least-squares sense) to this problem would be to find α_i and β_j such that

$$\sum_{k=1}^{n} \left| H_k - \frac{\sum_{i=1}^{N} \alpha_i f_i(s_k)}{\sum_{j=1}^{D} \beta_j g_j(s_k)} \right|^2 \tag{5}$$

is minimized. Common practice solving such problems is to linearize the problem to get the following expression:

$$\sum_{k=1}^{n} \left| \sum_{i=1}^{N} \alpha_i f_i(s_k) - H_k \sum_{j=1}^{D} \beta_j g_j(s_k) \right|^2. \tag{6}$$

Equation (6) is linear in its unknowns α_i and β_j and can be solved using standard least-squares techniques. Unfortunately,

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the solutions of this linear problem are not the same as those of the original problem. Note that the weighting factor of

$$\frac{1}{\left|\sum_{j=1}^{D}\beta_j g_j(s_k)\right|^2} \tag{7}$$

was dropped in (6).

Now we propose an iterative scheme. Suppose we have already found some $\alpha_i^{(1)}$ and $\beta_j^{(1)}$ that minimize (6). This result can be used to start the iterative algorithm that minimizes following *weighted* expressions (for $t = 1, 2, \ldots$)

$$\sum_{k=1}^{n}\frac{1}{\left|\sum_{j=1}^{D}\beta_j^{(t)} g_j(s_k)\right|^2} \times \left|\sum_{i=1}^{N}\alpha_i^{(t+1)} f_i(s_k) - H_k\sum_{j=1}^{D}\beta_j^{(t+1)} g_j(s_k)\right|^2. \tag{8}$$

Each iteration tries to correct the missing weighting factor in (6) by using the denominator of the previous iteration.

This approach is known as the Sanathanan–Koerner (SK) iteration [6] in the s-domain for continuous time systems or the Steiglitz–McBride iteration [7] in the z-domain for discrete time systems. These iterative techniques are extensively used, and their properties have been studied. For more information, we refer to [8] and [9]. The VF methodology also can be seen as an implementation of the iteration described in this section, as we will see in the next section.

IV. Closer Look at Vector Fitting

We now take a closer look at the VF iteration [1]. In each iteration, both $\sigma(s)f(s)$ and $\sigma(s)$ are approximated by rational functions using a common set of fixed poles $\overline{a}_i$

$$\sigma(s)f(s) = \sum_{i=1}^{D}\frac{c_i^{(1)}}{s-\overline{a}_i} + L^{(1)}(s)$$
$$\sigma(s) = \sum_{i=1}^{D}\frac{c_i^{(2)}}{s-\overline{a}_i} + L^{(2)}(s) \tag{9}$$

with $L^{(1)}(s) = d + s\,h$ and $L^{(2)}(s) = 1$.

Based on (2) and (3), we can easily perform a basis transformation, or do a so-called "pole-relocation," for each equation in (9)

$$\begin{aligned}&\frac{\prod_{i=1}^{D}(s-\overline{a}_i)}{\prod_{i=1}^{D}(s-\overline{z}_i)}\left[\sum_{i=1}^{D}\frac{c_i^{(l)}}{s-\overline{a}_i} + L^{(l)}(s)\right]\\ &= \frac{\prod_{i=1}^{D}(s-\overline{a}_i)}{\prod_{i=1}^{D}(s-\overline{z}_i)}\frac{P^{(l)}(s)}{\prod_{i=1}^{D}(s-\overline{a}_i)}\\ &= \frac{P^{(l)}(s)}{\prod_{i=1}^{D}(s-\overline{z}_i)} = \sum_{i=1}^{D}\frac{d_i^{(l)}}{s-\overline{z}_i} + L'^{(l)}(s)\end{aligned} \tag{10}$$

for some coefficients $d_i^{(l)}$ and a polynomial $L'^{(l)}(s)$.

Multiplying the second equation in (9) by H_k, evaluating in s_k and equating both equations (as in [1]), and using (10), we get the system of equations

$$\frac{\prod_{i=1}^{D}(s_k-\overline{a}_i)}{\prod_{i=1}^{D}(s_k-\overline{z}_i)}\left[\sum_{i=1}^{D}\frac{c_i^{(1)}}{s_k-\overline{a}_i} + L^{(1)}(s_k) - H_k\sum_{i=1}^{D}\frac{c_i^{(2)}}{s_k-\overline{a}_i} - L^{(2)}(s_k)H_k\right] = 0 \tag{11}$$

if and only if

$$\sum_{i=1}^{D}\frac{d_i^{(1)}}{s_k-\overline{z}_i} + L'^{(1)}(s_k) - H_k\sum_{i=1}^{D}\frac{d_i^{(2)}}{s_k-\overline{z}_i} = L'^{(2)}(s_k)H_k \tag{12}$$

for all sample points s_k.

The first set of (11) is the same as in VF for poles $\overline{a}_i$, except for the weighting factor

$$W(s) = \frac{\prod_{i=1}^{D}(s-\overline{a}_i)}{\prod_{i=1}^{D}(s-\overline{z}_i)}. \tag{13}$$

The second set of (12) is exactly the VF formulation for a set of poles $\overline{z}_i$.

Suppose now that we have solved the VF pole identification system with starting poles $\overline{a}_i$. Expanding $\sigma(s)$ in its poles and zeros gives rise to an expression that matches $1/W(s)$ exactly if $\overline{z}_i$ is chosen to match the zeros of σ. In that case, solving the weighted problem (11) obtains the same approximation as solving (12). The last is exactly the second iteration in the VF methodology, while the first exactly matches the iterative scheme of Section III with $f_i(s) = g_i(s)$ equal to the basis function specified in Section II and $f_{D+1}(s) = g_{D+1}(s) = 1$ and $f_{D+2}(s) = s$.

V. Conclusion

VF is identified as an elegant reformulation of the SK iteration. The pole-based basis that is used in VF has numerical advantages over the power-series approach, which is still often used in other implementations of the SK iteration. Powers of s (which lead to ill-conditioning) are avoided in the linear system by using the pole-based basis.

Furthermore, the explicit pole representation of it makes the enforcement of stable poles easily achievable (by flipping the unstable poles) as compared to other implementations of the SK iteration.

Note that the SK iteration is known not to exactly minimize a least-squares distance. However, it provides good models and remains stable during successive iterations [9], [10]. Theoretically, both the VF and SK methodologies should compute the same solution to the problem. In practice, VF produces favorable linear systems that do not contain high powers of s as in a direct implementation of SK.

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