CAUSALITY DETERMINATION AND TIME DELAY EXTRACTION BY MEANS OF THE EIGENFUNCTIONS OF THE HILBERT TRANSFORM

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Abstract

Frequency domain causality is determined by means of the Hilbert transform relating the imaginary part to the real part of the transfer function. Projecting on the eigenfunctions of the Hilbert transform leads to a simple and easily understood causality determination algorithm for measured (sampled) transfer functions. The causality determination algorithm can then easily be transformed into a delay estimation algorithm, by introducing a linearly varying phase factor. Several examples, both analytical and numerical, demonstrate the strength and accuracy of the proposed algorithm.

1 INTRODUCTION

Time delay identification and extraction is an important signal processing problem with applications in many areas, such as radar, sonar [1], wireless communications [2] and transmission line modelling [3]. The link with causality and the Hilbert transform [4, 5] is straightforward, since time delays can pull a non-causal signal into the causal region or conversely pull a causal signal into the non-causal region. Several algorithms have been introduced to extract delays in the frequency domain, based on the Hilbert transform [6, 7] or the cepstrum [8], while delayed rational functions [9] were used to extract delays in the time domain. In this contribution, we determine causality in the frequency domain by using a classic argument [4] which says that the imaginary part of the transfer function must be the Hilbert transform of the real part of the transfer function. Projecting onto the well-known rational eigenfunctions of the Hilbert transform [5], which happen to be closely related to the scaled Laguerre functions [10, 11], leads to a simple and easily understood causality determination algorithm for measured (sampled) transfer functions. This same causality determination algorithm can then easily be transformed into a delay estimation algorithm, by introducing a linearly varying phase factor. Several examples, both analytical and numerical, demonstrate the strength and accuracy of the proposed algorithm.

2 THE HILBERT TRANSFORM

In what follows we consider real $L_2(\mathbb{R})$ time domain functions $f(t)$ and their Fourier transform (transfer function)

$$F(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\, dt \tag{1}$$

The function $f(t)$ is causal if $f(t) = 0$ whenever $t < 0$. The Hilbert transform [5] of the frequency domain function $F(\omega)$ is defined as

$$\mathcal{H}\{F\}(\omega) = \frac{1}{\pi}\,\mathrm{PV}\int_{-\infty}^{\infty} \frac{F(\omega')}{\omega' - \omega}\, d\omega' \tag{2}$$

It is well known [4] p. 251 that when a function $f(t)$ is causal, then the imaginary part of the transfer function is the Hilbert transform of the real part, i.e.,

$$\Im F(\omega) = \mathcal{H}\{\Re F\}(\omega) \tag{3}$$

In [5] it is shown that the basis functions

$$\phi_n(\omega) = \frac{(1+i\omega)^n}{(1-i\omega)^{n+1}} \qquad n \in \mathbb{Z} \tag{4}$$

which form a complete orthogonal basis for $L_2(\mathbb{R})$ with in particular

$$\int_{-\infty}^{\infty} \phi_n(\omega)\overline{\phi_m(\omega)}\, d\omega = \pi\delta_{n,m} \tag{5}$$

are such that

$$\mathcal{H}\{\phi_n\}(\omega) = i\,\mathrm{sgn}(n)\,\phi_n(\omega) \tag{6}$$

with $\mathrm{sgn}(0) = 1$. This means that the basis functions $\{\phi_n(\omega)\}$ are the eigenfunctions of the Hilbert transform with eigenvalues $\pm i$. It is easy to show that the same is also true for the scaled basis $\{\phi_n(\omega/L)\}$ with $L > 0$, with of course the normalization constant $L\pi$ instead of π in the r.h.s. of equation (5). Note that, with the convenient change of variables

$$e^{i\theta} = \frac{1+i\omega/L}{1-i\omega/L} \quad \Rightarrow \quad \omega/L = \tan\left(\frac{\theta}{2}\right) \tag{7}$$

the basis functions can be written in Fourier-series fashion as

$$\phi_n(\omega/L) = \cos\left(\frac{\theta}{2}\right) e^{i(n+1/2)\theta} \tag{8}$$

which at the same time proves that $\Re\phi_n(\omega/L)$ resp. $\Im\phi_n(\omega/L)$ are odd resp. even functions of ω. Expanding the Fourier transform $F(\omega)$ of a real function $f(t) \in L_2(\mathbb{R})$ as

$$F(\omega) = \sum_{n=-\infty}^{\infty} a_n \phi_n(\omega/L) \tag{9}$$

it is seen that the coefficients

$$a_n = \frac{1}{\pi L}\int_{-\infty}^{\infty} F(\omega)\overline{\phi_n(\omega)}\, d\omega \tag{10}$$

are real, since $\Re F(\omega)$ is even and $\Im F(\omega)$ is odd. Utilizing the fact that $\overline{\phi_n(\omega/L)} = \phi_{-n-1}(\omega/L)$, we can write

$$\Re F(\omega) \;=\; \frac{1}{2}\sum_{n=-\infty}^{\infty}(a_n + a_{-n-1})\phi_n(\omega/L) \tag{11}$$

$$\Im F(\omega) \;=\; \frac{1}{2i}\sum_{n=-\infty}^{\infty}(a_n - a_{-n-1})\phi_n(\omega/L) \tag{12}$$

For a causal function we need the imaginary part $\Im F(\omega)$ to be the Hilbert transform of the real part $\Re F(\omega)$. By inspection this will be the case when

$$(a_n + a_{-1-n})\,\mathrm{sgn}\,(n) = -(a_n - a_{-1-n}) \quad \forall n \in \mathbb{Z} \tag{13}$$

in other words when $a_n = 0$ for $n \geq 0$. Hence a causal function must have a frequency domain expansion

$$F(\omega) = \sum_{n=-\infty}^{-1} a_n \phi_n(\omega/L) = \sum_{n=0}^{\infty} a_{-n-1} \overline{\phi_n(\omega/L)} \tag{14}$$

In other words, the real and imaginary part of $F(\omega)$ are related by

$$\Re F(\omega) = \sum_{n=0}^{\infty} a_{-n-1} \Re \phi_n(\omega/L) \tag{15}$$

$$\Im F(\omega) = -\sum_{n=0}^{\infty} a_{-n-1} \Im \phi_n(\omega/L) \tag{16}$$

where the coefficients a_{-n-1} are real. This provides an easy to implement approximate causality test in the frequency domain when, as is often the case, there are discrete frequency samples of the transfer function available, i.e., $F(\omega_k)$ is given on the discrete frequency array $0 \leq \omega_1 \leq \omega_2 \leq \ldots \leq \omega_M$. We therefore solve the following generally over-determined systems ($M \geq N$) in a least squares sense :

$$\Re F(\omega_k) = \sum_{n=0}^{N-1} \alpha_n \Re \phi_n(\omega_k/L) \quad k = 1, \ldots, M \tag{17}$$

$$-\Im F(\omega_k) = \sum_{n=0}^{N-1} \beta_n \Im \phi_n(\omega_k/L) \quad k = 1, \ldots, M \tag{18}$$

If the transfer function is causal, the vectors α and β must be the same. To assess the difference, we propose to minimize the approximate 'causality' measure

$$\mathcal{C}(\alpha, \beta) = \frac{\sqrt{\sum_{n=0}^{N-1} (\alpha_n - \beta_n)^2}}{\sqrt{2\left(\sum_{n=0}^{N-1} (\alpha_n^2 + \beta_n^2)\right)}} \tag{19}$$

which is such that $0 \leq \mathcal{C}(\alpha, \beta) \leq 1$, and $\mathcal{C} = 0$ means utter causality.

3 APPLICATION TO DELAY ESTIMATION

The idea is the following: suppose that the real function $f(t)$ is non-vanishing only from the time $T_0 \geq 0$ on, and our intention is to identify (or estimate) the delay (or offset) T_0. It is easy to see that the Fourier transform $F(\omega)$ is

$$\begin{aligned} F(\omega) &= \int_{t=T_0}^{\infty} e^{-i\omega t} f(t)\, dt \\ &= e^{-i\omega T_0} \int_{t=0}^{\infty} e^{-i\omega t} f(t + T_0)\, dt \\ &= e^{-i\omega T_0} G(\omega) \end{aligned} \tag{20}$$

where $G(\omega)$ is the Fourier transform of a causal function. In other words, when $0 \leq T \leq T_0$, the transfer function $F(\omega)e^{i\omega T}$ is causal, and when $T > T_0$, the transfer function $F(\omega)e^{i\omega T}$ contains a non-causal part. Since the phase ωT is modulo 2π, we can restrict our investigations to the case

$$T \leq T_{\max} = \frac{2\pi}{\omega_M} \tag{21}$$

For each potential delay $0 \leq T \leq T_{\max}$, we solve the following over-determined systems in a least squares sense :

$$\sum_{n=0}^{N-1} \alpha_n \Re \phi_n(\omega_k/L) = \Re \left[F(\omega_k) e^{i\omega T} \right] \tag{22}$$

$$\sum_{n=0}^{N-1} \beta_n \Im \phi_n(\omega_k/L) = -\Im \left[F(\omega_k) e^{i\omega T} \right] \tag{23}$$

It should be noted that the $M \times N$ matrices $\Re \phi_n(\omega_k/L)$ and $\Im \phi_n(\omega_k/L)$ in the l.h.s. do not depend on T, and hence a linear solver admitting multiple r.h.s. should be the option, where each T in the admissible range generates the r.h.s. $\Re \left[F(\omega_k) e^{i\omega T} \right]$ or $-\Im \left[F(\omega_k) e^{i\omega T} \right]$. The solutions α and β are therefore functions of T, and hence also the causality measure $\mathcal{C}(\alpha, \beta) = \mathcal{C}(T)$. As a last problem there is the choice for the parameter L. In our simulations we always take the maximum angle $\theta_M = \pi \frac{N-1}{N} < \pi$, resulting in

$$L = \frac{\omega_M}{\tan\left(\frac{\pi}{2} \frac{N-1}{N}\right)} \tag{24}$$

which is somewhat larger than the available bandwidth ω_M. Also, in all the examples the number N of Hilbert eigenfunctions has been fixed at $N = 20$.

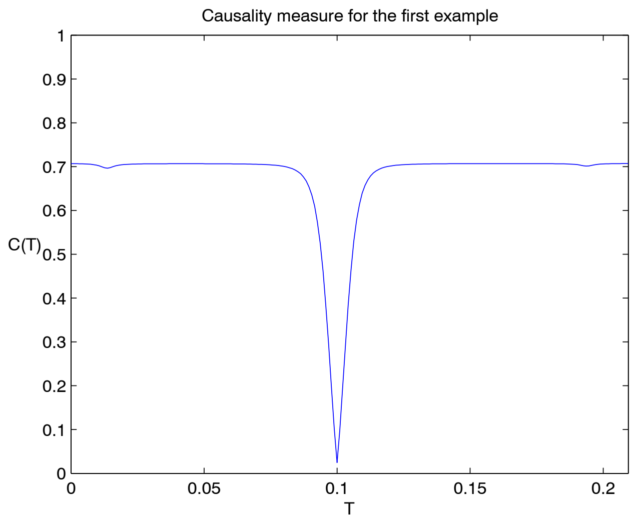

Figure 1: Causality measure for the two-pole example.

4 EXAMPLES

4.1 TWO POLE EXAMPLE

Here we model a two-pole example with transfer function

$$F(\omega) = e^{-i\omega T_0}\left(\frac{r}{i\omega + p} + \frac{\overline{r}}{i\omega + \overline{p}}\right) \tag{25}$$

where $T_0 = 0.1$, $r = 1 + 3i$, $p = 1 + 2i$. The data consist of $M = 200$ equispaced samples in $[\omega_1 = 0, \omega_M = 30]$. The causality measure is plotted in Fig. 1. The causality measure has a unique minimum at $T = 0.09998$, which is pretty close to T_0.

4.2 DAWSON'S INTEGRAL EXAMPLE

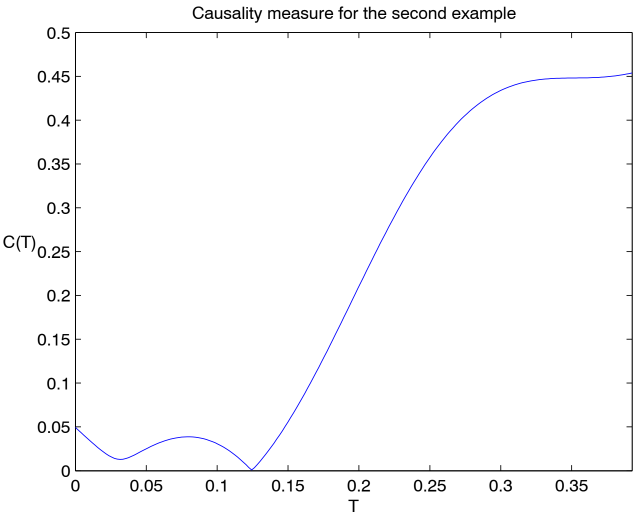

Figure 2: Causality measure for the Dawson example.

Here we model a more complex analytic example

$$F(\omega) = e^{-i\omega T_0}\left(e^{-\omega^2} - \frac{2i}{\sqrt{\pi}} D(\omega)\right) \tag{26}$$

where $D(x)$ is Dawson's integral

$$D(x) = e^{-x^2}\int_0^x e^{t^2}\,dt = \frac{\sqrt{\pi}}{2} e^{-x^2}\,\mathrm{erfi}\,(x) \tag{27}$$

and $\mathrm{erfi}\,(x)$ is the so-called imaginary error function. Since $-2D(\omega)/\sqrt{\pi}$ is the Hilbert transform of $e^{-\omega^2}$ [5], it is seen that $F(\omega)$ is the Fourier transform of a causal function delayed with offset T_0. In the present example we take $T_0 = 0.125$, and the data consist of $M = 200$ equispaced samples in $[\omega_1 = 0, \omega_M = 16]$. The causality measure is plotted in Fig. 2. The causality measure exhibits two local minima, with a global minimum at $T = 0.12432$, which is close to T_0.

4.3 MEASURED BANDPASS FILTER EXAMPLE

As a last example we take measured data of the transmission coefficient $S_{12}(\omega)$ of a microwave bandpass filter. The magnitude and phase are shown in Fig. 3. The decreasing behavior of the phase indicates there is a big chance that a delay T_0

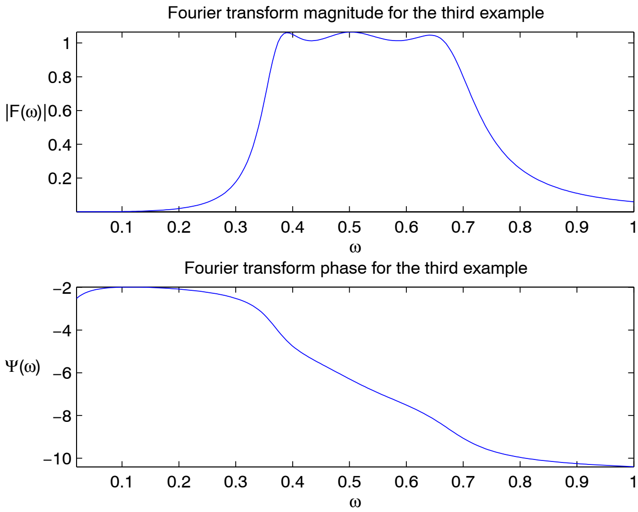

Figure 3: Magnitude and phase (in rad.) of the bandpass filter.

is present. If we estimate the slope of the phase with linear regression we find

$$\Psi(\omega) \approx c_1 + c_2\omega \tag{28}$$

with $c_1 = -0.4191$ and $c_2 = -11.2291$. A crude estimate of T_0 is therefore

$$T_0 \approx c_2 - \frac{2\pi}{\omega_M}\left\lfloor \frac{\omega_M c_2}{2\pi} \right\rfloor = 1.3373 \tag{29}$$

The causality measure is plotted in Fig. 4. It is seen that the causality measure has a unique minimum at $T = 1.13666$, which is therefore our final estimated offset T_0.

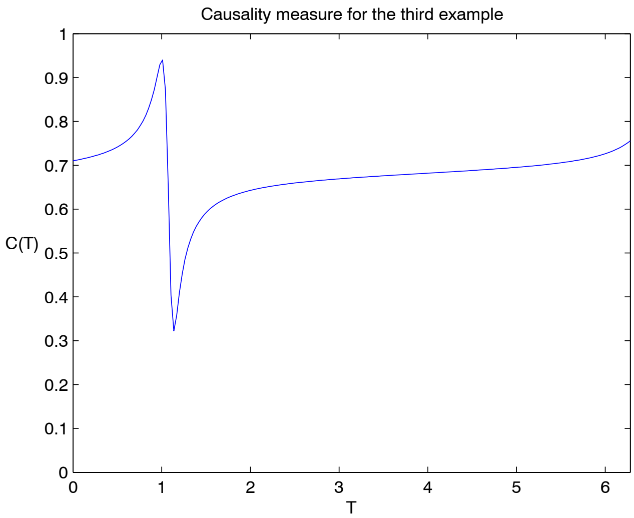

Figure 4: Causality measure for the bandpass filter example.

5 CONCLUSION

The astute reader will have recognized the close relationship between the rational Hilbert transform eigenfunctions and the scaled Laguerre functions [10, 11]

$$\Psi_n(\omega) = \frac{\sqrt{2\gamma}}{i\omega + \gamma} \left(\frac{i\omega - \gamma}{i\omega + \gamma} \right)^n \quad n = 0, 1, \ldots \tag{30}$$

as it is easy to verify that

$$\Psi_n(\omega) = \sqrt{\frac{2}{\gamma}} (-1)^n \overline{\phi_n(\omega/\gamma)} \tag{31}$$

It is seen that the time scale factor γ and the frequency scaling factor L are the same. However, the theory leading to the expression for the optimal time scale factor [10] in the time domain, i.e.,

$$\gamma = \sqrt{\int_0^\infty t f'(t)^2 \, dt \Big/ \int_0^\infty t f(t)^2 \, dt} \tag{32}$$

is at first sight of no immediate help, since we are working with sampled data in the frequency domain. Hence additional study in that direction should be pursued. To conclude and pave the way for further research it would also be exciting if, instead of the scaled Laguerre basis, one would be able to use the much more general and powerful Kautz basis [11], in order to obtain still more performing causality and delay estimation algorithms.

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